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Numerical calculation of the mutual coherence function for undulator radiation in storage rings

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THE GOAL

Our ultimate goal is to describe statistical properties (partial coherence) of synchrotron radiation along a beamline for beamline design purposes on desktop computers.

THEORY: MUTUAL COHERENCE FUNCTION

The mutual coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2)$ describes second order correlation completely.

From the mutual coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2)$ the mean intensity can be deduced.

The mutual coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2)$

- can be measured with a double slit experiment (Young’s experiment)
- is related to the Wigner function through a Fourier transform:

$$B(\boldsymbol{r}, \boldsymbol{\theta}) = \int \Gamma\left(\boldsymbol{r} + \frac{\boldsymbol{u}}{2}\right)\left(\boldsymbol{r} - \frac{\boldsymbol{u}}{2}\right) \exp(ik\boldsymbol{\theta}\boldsymbol{u})\, d\boldsymbol{u}$$

Consider the free space formula for the wavefront propagation using Huygens-Fresnel propagator:

$$E'(\boldsymbol{r}) = -i\frac{k}{2\pi}\int E(\boldsymbol{r}')\frac{e^{ikR}}{R}\cos(\alpha)d\boldsymbol{r}'$$

With:

$R(\boldsymbol{r},\boldsymbol{r}') = \|\boldsymbol{r}-\boldsymbol{r}'\|$ and inclination factor $\cos(\alpha)$.

A similar formula exists for the propagation of the mutual coherence function:

$$\Gamma'(\boldsymbol{r}_1,\boldsymbol{r}_2) = \left(\frac{k}{2\pi}\right)^2\int \Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2)\frac{e^{ik(R_2-R_1)}}{R_1R_2}\cos(\alpha_1)\cos(\alpha_2)d\boldsymbol{r_1}'\,d\boldsymbol{r_2}'$$

Mind however, that the integral is **4d**.

THEORY: BRIGHTNESS CONVOLUTION THEOREM

The brightness convolution theorem is an approximate way to calculate the Wigner function:

$$B(\boldsymbol{r},\boldsymbol{\theta}) = N_e \int B_0(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{\theta}-\boldsymbol{\theta}')f(\boldsymbol{r}',\boldsymbol{\theta}')d\boldsymbol{r}'d\boldsymbol{\theta}'$$

We can obtain the mutual coherence function with a Fourier transformation:

$$\Gamma\left(\boldsymbol{r}+\frac{\boldsymbol{u}}{2},\boldsymbol{r}-\frac{\boldsymbol{u}}{2}\right) = \int B(\boldsymbol{r},\boldsymbol{\theta})\exp(-ik\boldsymbol{\theta}\boldsymbol{u})\,d\boldsymbol{u}$$

The theorem assumes that:

1. different electrons of the electron beam are statistically independent.
2. The variation of the magnetic guide field across the electron beam dimension is **negligible**.

The electron phase space density can be described in terms of the general 6x6 covariance matrix Σ:

$$f(\boldsymbol{r}',\boldsymbol{\theta}') = C\cdot\exp\left(-\mathbf{x}^{\mathrm{T}}\Sigma\boldsymbol{x}\right) \qquad \boldsymbol{x} = (x,x',y,y',\delta,z)$$

THE STRATEGY

Calculate $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2)$
(approximation: brightness convolution)

↓

Propagate $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2)$ along the beamline:
$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2) \rightarrow \Gamma'(\boldsymbol{r}_1, \boldsymbol{r}_2)$
Slow 4d integral for each optical element

For the application of the brightness convolution theorem we need a reference electric field E_0.

$$B_0(\boldsymbol{r},\boldsymbol{\theta}) = \int \mathrm{E}_0^*\left(\boldsymbol{r}+\frac{\boldsymbol{u}}{2}\right) E_0\left(\boldsymbol{r}-\frac{\boldsymbol{u}}{2}\right)\exp(ik\boldsymbol{\theta}\boldsymbol{u})\, d\boldsymbol{u}$$

We calculate the reference electric field E_0 with SRW.

NUMERICS: COHERENT MODE DECOMPOSITION

One can show that the mutual coherence function can always be represented in coherent modes:

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \beta_n \phi_n^*(\boldsymbol{r}_1)\phi_n(\boldsymbol{r}_2)$$

The $\phi_n(\boldsymbol{r})$ are called coherent modes and they are orthogonal. The eigenvalues β_n can be interpreted as mode intensities.

We perform this decomposition numerically.

NUMERICS: PROPAGATION OF THE MUTUAL COHERENCE FUNCTION

Putting the decomposed form into the propagation formula we find:

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = \left(\frac{k}{2\pi}\right)^2 \int \boldsymbol{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2)\frac{e^{ik(R_2-R_1)}}{R_1R_2}\cos(\alpha_1)\cos(\alpha_2)d\boldsymbol{r_1}'\, d\boldsymbol{r_2}'$$

$$= \left(\frac{k}{2\pi}\right)^2 \int \sum_n \beta_n \phi_n^*(\boldsymbol{r}_1)\phi_n(\boldsymbol{r}_2)\frac{e^{ik(R_2-R_1)}}{R_1R_2}\cos(\alpha_1)\cos(\alpha_2)d\boldsymbol{r_1}'\, d\boldsymbol{r_2}'$$

$$= \left(\frac{k}{2\pi}\right)^2 \sum_{\boldsymbol{n}} \beta_n \left(\int \phi_n^*(\boldsymbol{r}_1)\,\frac{e^{-ikR_1}}{R_1}\cos(\alpha_1)d\boldsymbol{r_1}'\right)\left(\int \phi_n(\boldsymbol{r}_2)\frac{e^{ikR_2}}{R_2}\cos(\alpha_2)d\boldsymbol{r_2}'\right)$$

The modes are propagated like electric fields in normal wave optics.

NUMERICS: PROPAGATION OF THE MUTUAL COHERENCE FUNCTION II

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = \left(\frac{k}{2\pi}\right)^2 \sum_{\boldsymbol{n}}^{\boldsymbol{N}} \beta_n \left(\int \phi_n^*(\boldsymbol{r}_1) \frac{e^{-ikR_1}}{R_1} \cos(\alpha_1) d\boldsymbol{r_1}' \right) \left(\int \phi_n(\boldsymbol{r}_2) \frac{e^{ikR_2}}{R_2} \cos(\alpha_2) d\boldsymbol{r_2}' \right)$$

Assuming a **small number** of relevant modes we have a small number of 2d integrals per optical element opposed to a 4d integral per optical element.

We want to use SRW for the mode propagation.

THE STRATEGY

Calculate $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2)$
(approximation: brightness convolution)

↓

Perform coherent mode decomposition
(principal value analysis gives **N** modes)

↓

Propagate $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2)$ along the beamline:
$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2) \rightarrow \Gamma'(\boldsymbol{r}_1, \boldsymbol{r}_2)$
N times 2d integrals for each op. element

NUMERICS: TEST CASE: GAUSSIAN SCHELL MODEL

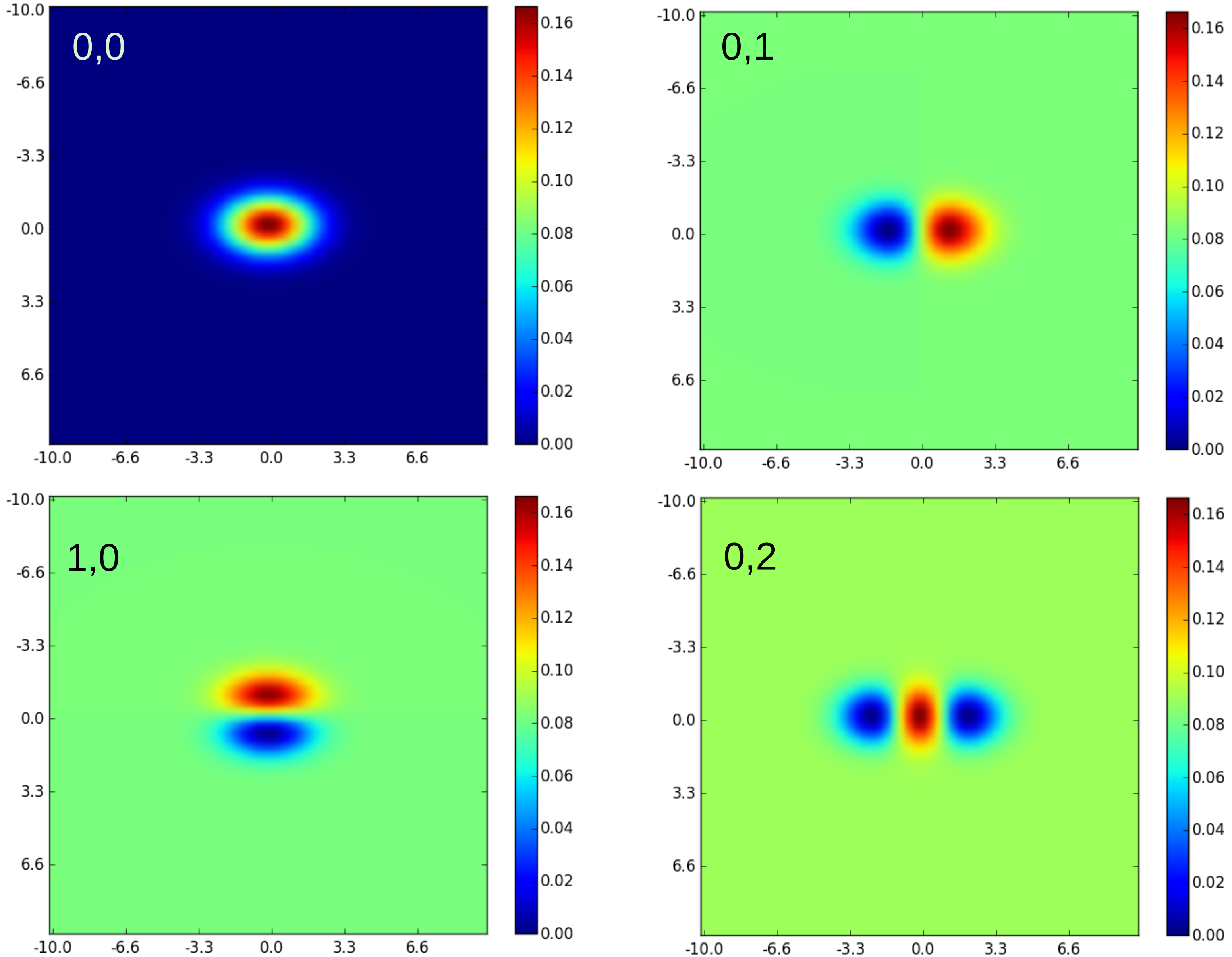
To **test** our decomposition algorithm we use a Gaussian Schell model cross spectral density for which analytical solution exists.

n	m	L2 norm of difference
0	0	2.64e-15
0	1	8.36e-15
1	0	8.58e-15
0	2	9.54e-15
1	1	1.17e-14
2	0	5.73e-15
0	3	1.28e-14
1	2	1.28e-14

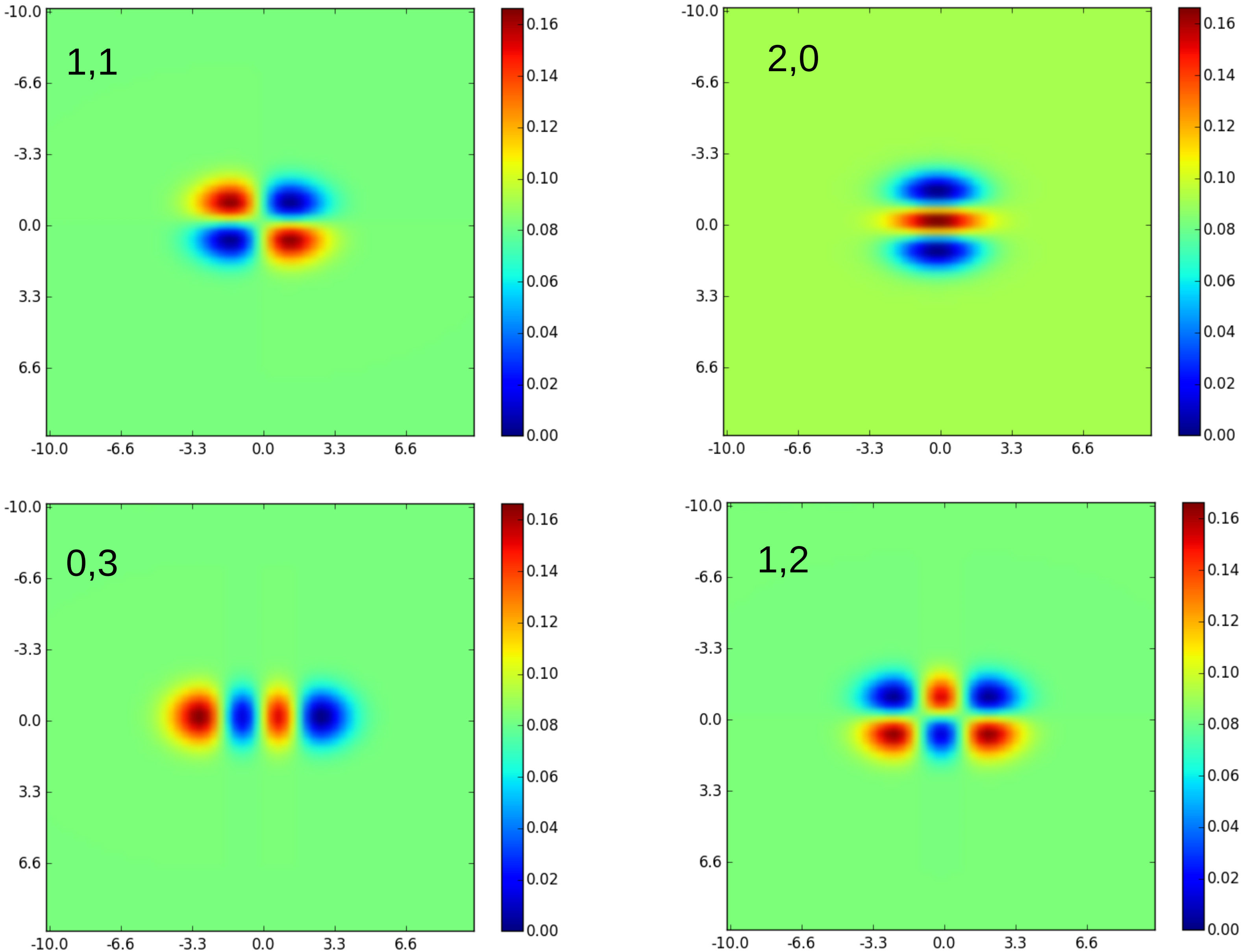
On an equidistant test grid with 60 points in x and 60 points in y direction
on the interval $\mathrm{I} = [-10,10]^2$ with:

$$\sigma_{sx} = 0.9 \qquad \sigma_{sy} = 1.5$$
$$\sigma_{gx} = 1.0 \qquad \sigma_{gy} = 1.5$$

The first 100 modes have a L_2 norm error below 1e-9.


0,0
0,1
1,0
0,2
-10.0
-6.6
-3.3
0.0
3.3
6.6
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00

NUMERICS: TESTCASE: GAUSSIAN SCHELL MODEL III



For a Gaussian reference electric field:

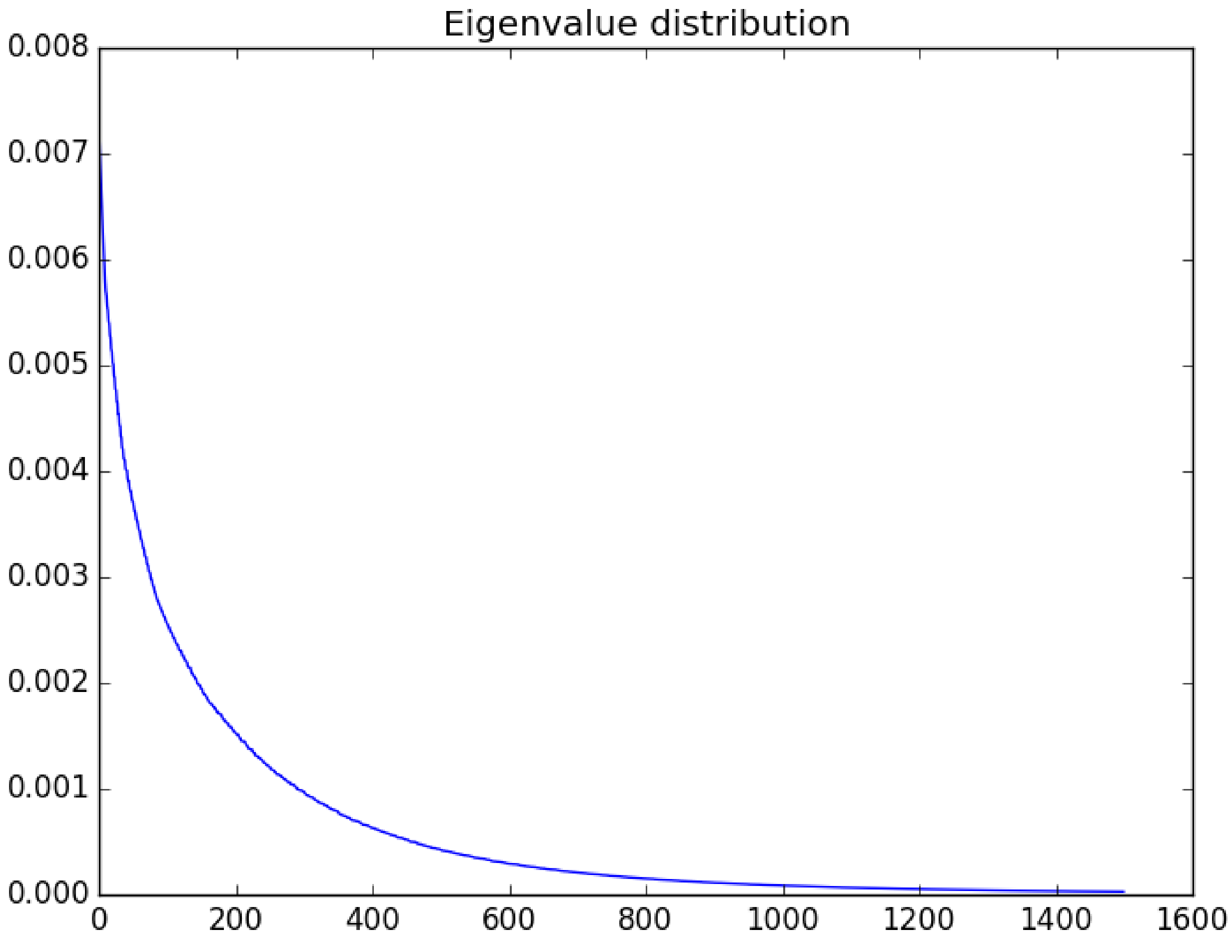
$$E_0(\boldsymbol{r}) = A \cdot \exp\left(-\frac{r^{\mathbf{2}}}{2\sigma^2}\right)$$

The mutual coherence function Γ can be calculated analytically.

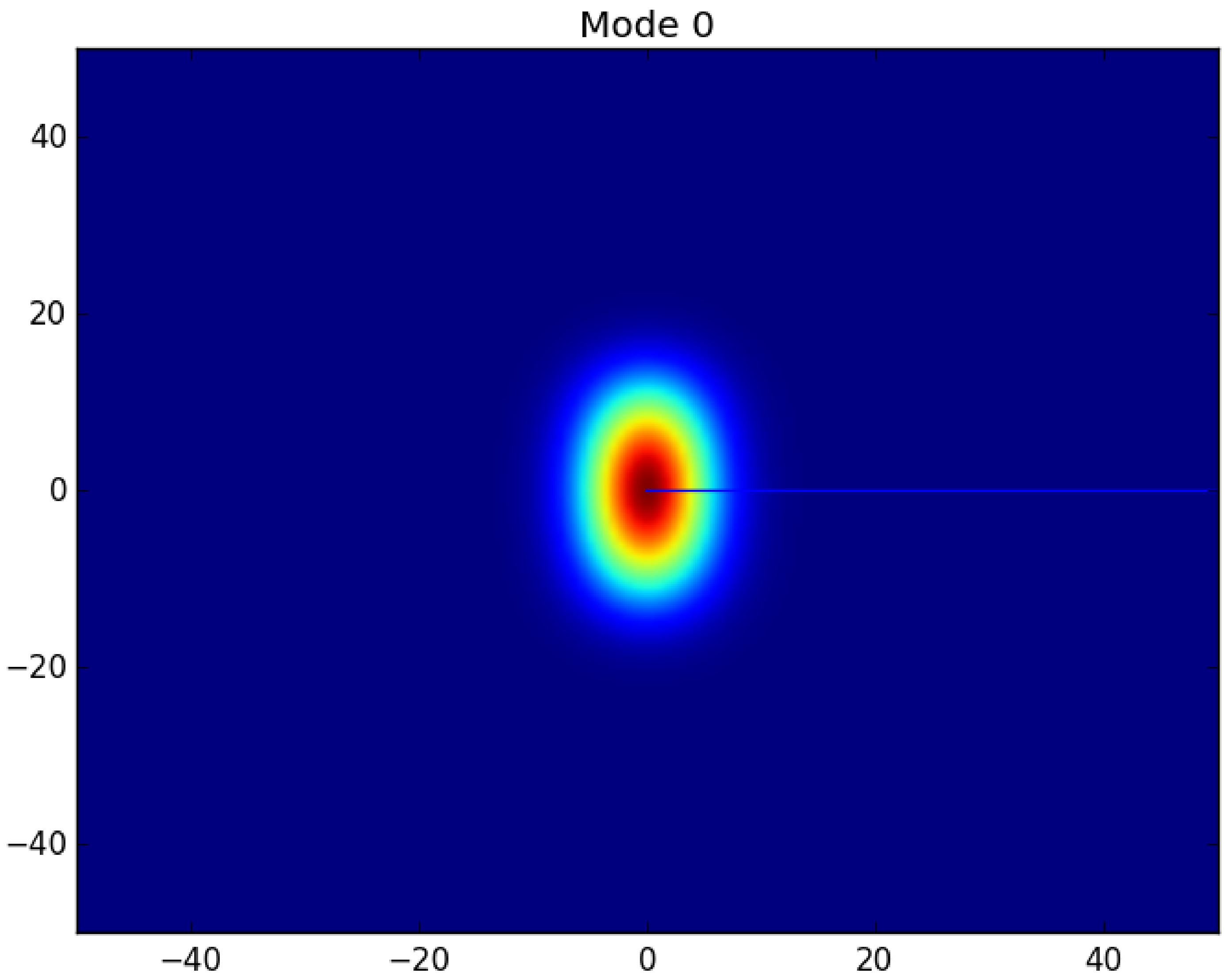
Tests of the implemented algorithm for such a test field against the analytical result have a difference in L_2 norm smaller than 1e-10.

NUMERICS: SOME COHERENT MODES

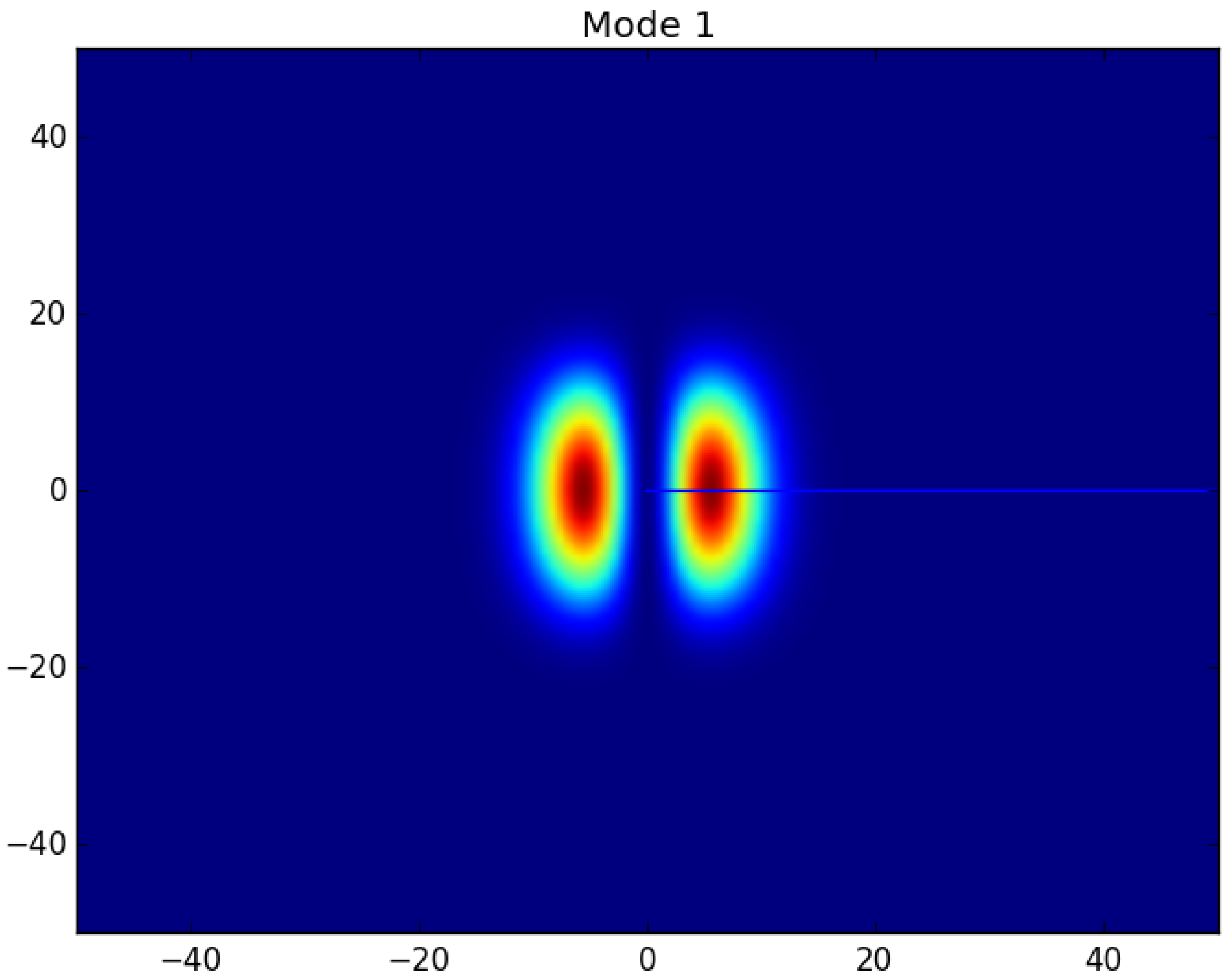
Coherent modes using the SRW example 10 undulator radiation as reference electric field E_0.



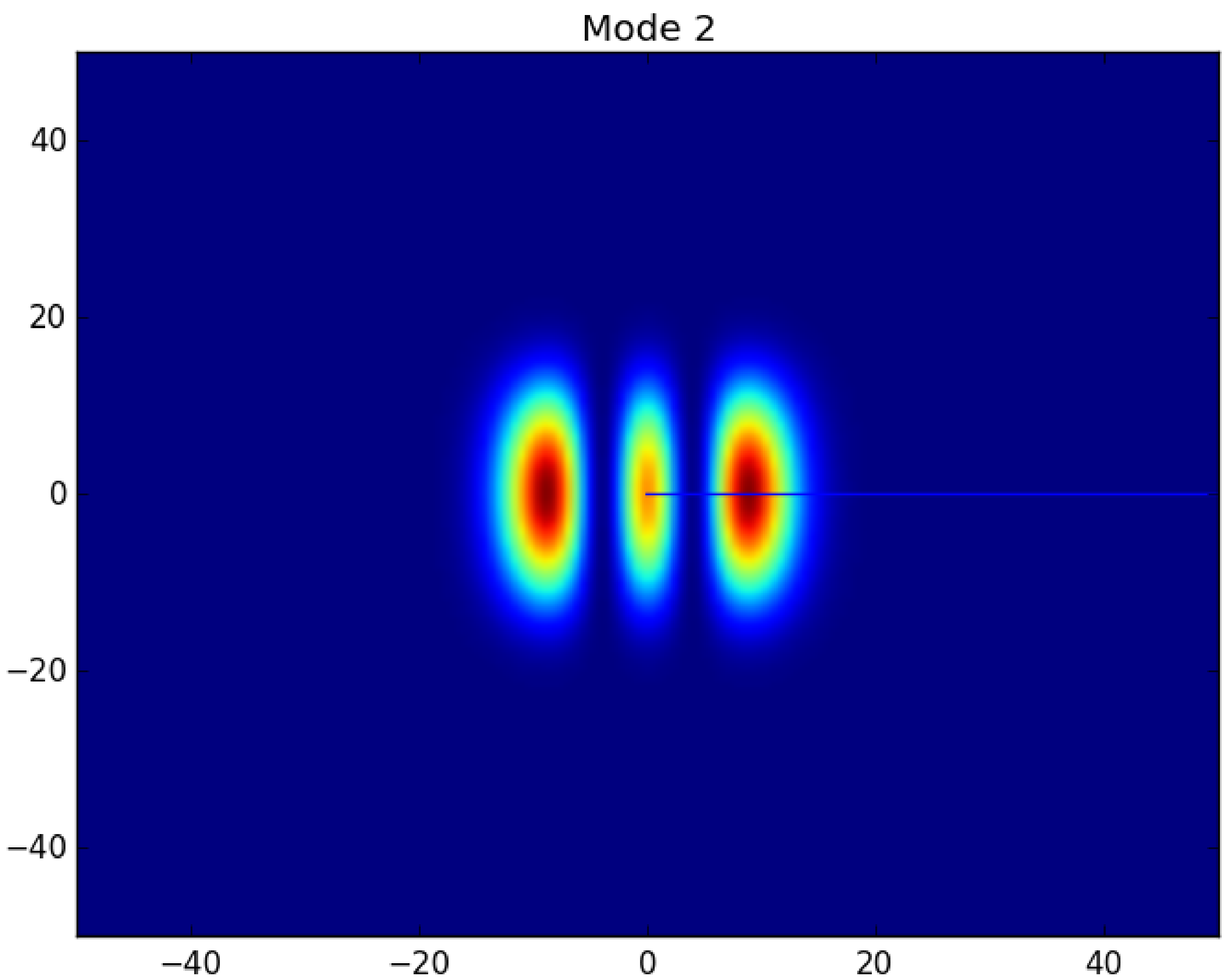
NUMERICS: SOME COHERENT MODES



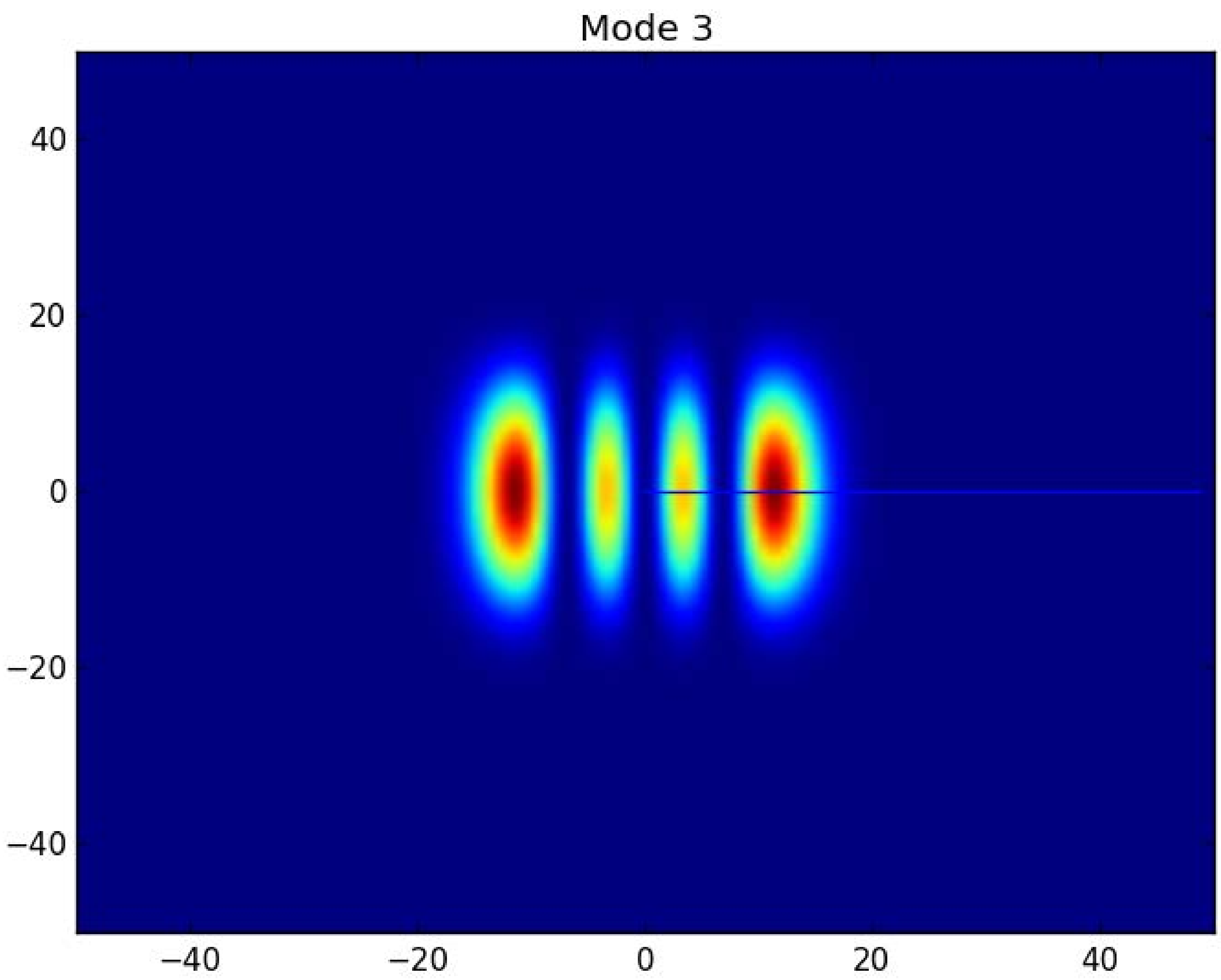
NUMERICS: SOME COHERENT MODES



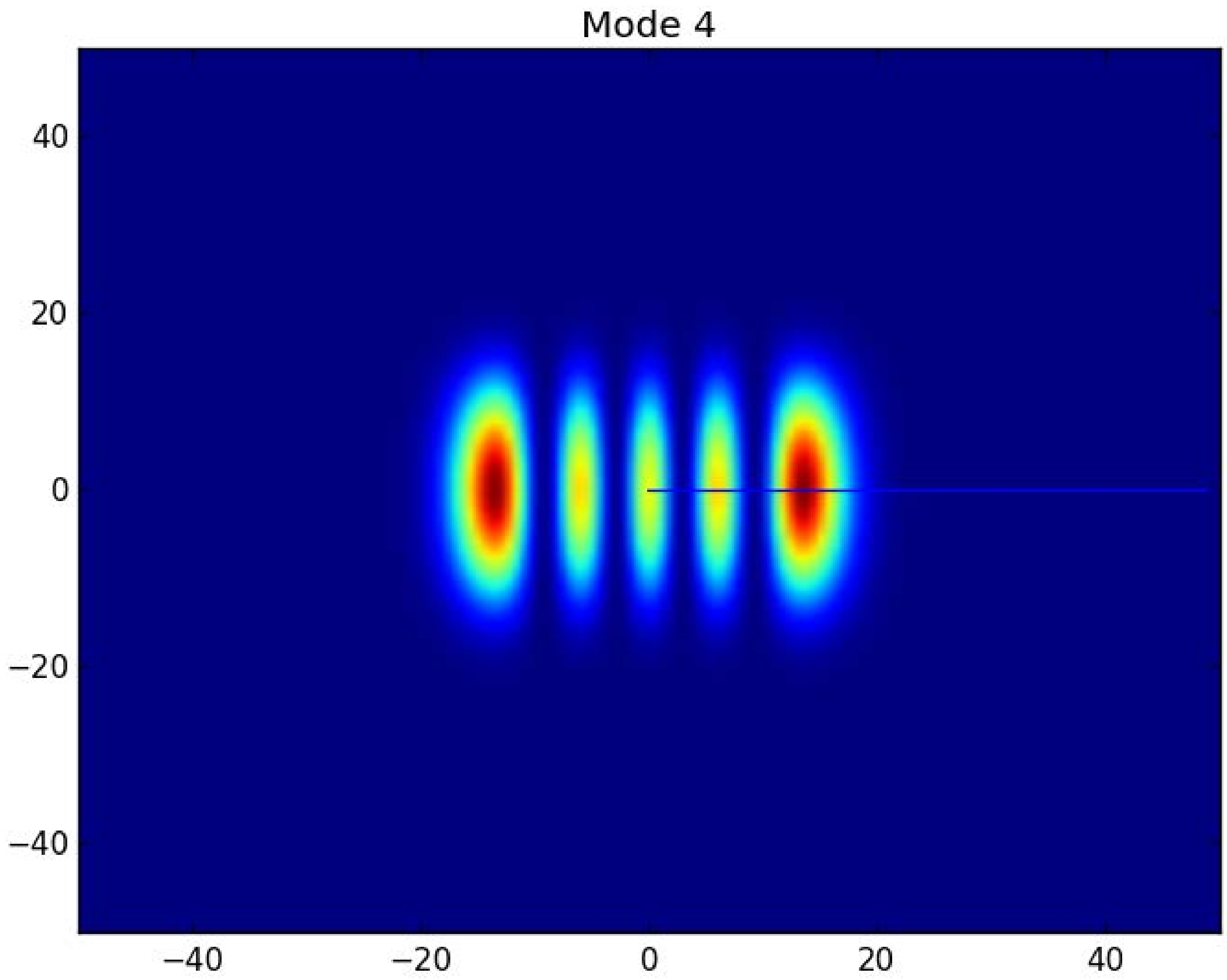
NUMERICS: SOME COHERENT MODES



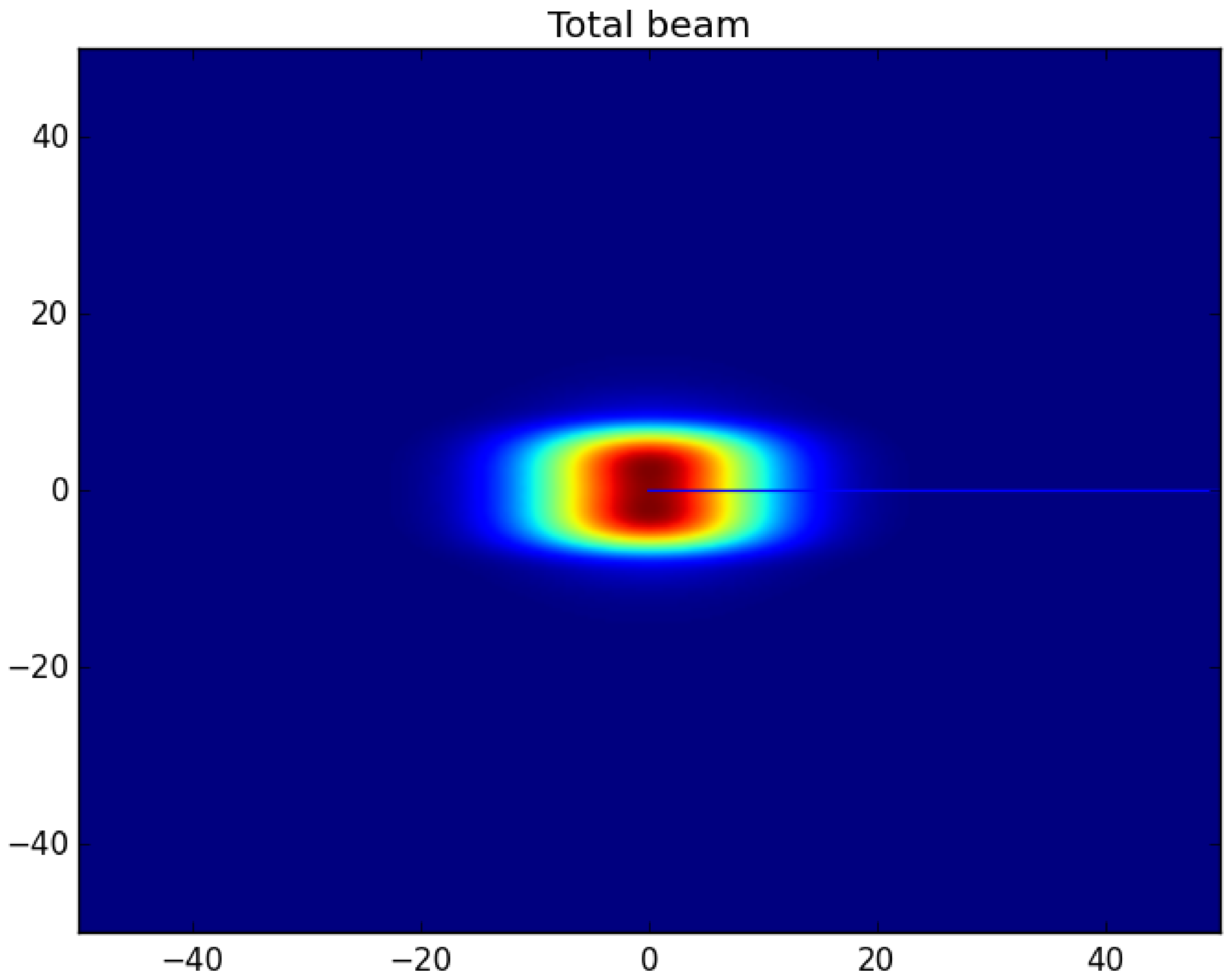
NUMERICS: SOME COHERENT MODES



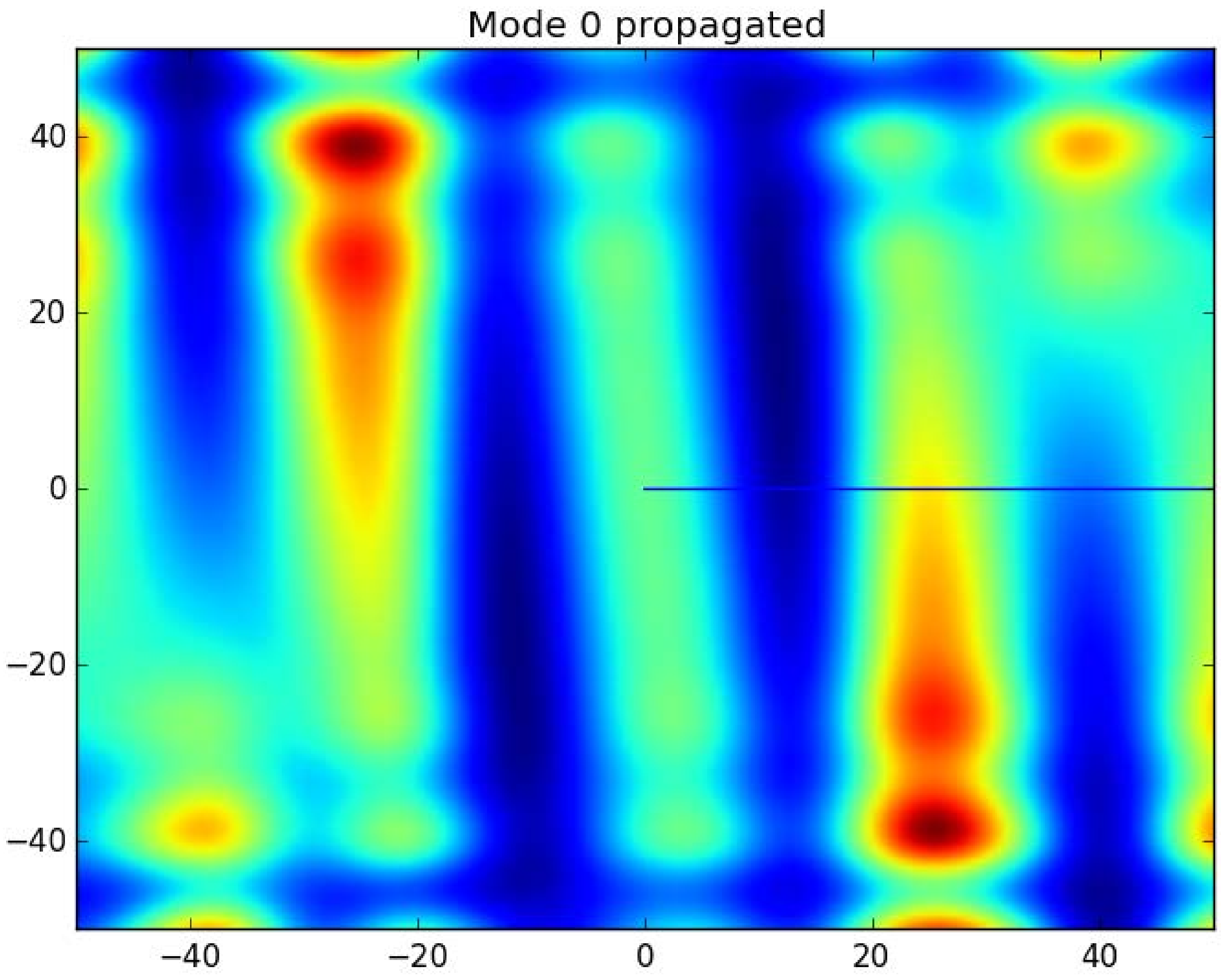
NUMERICS: SOME COHERENT MODES



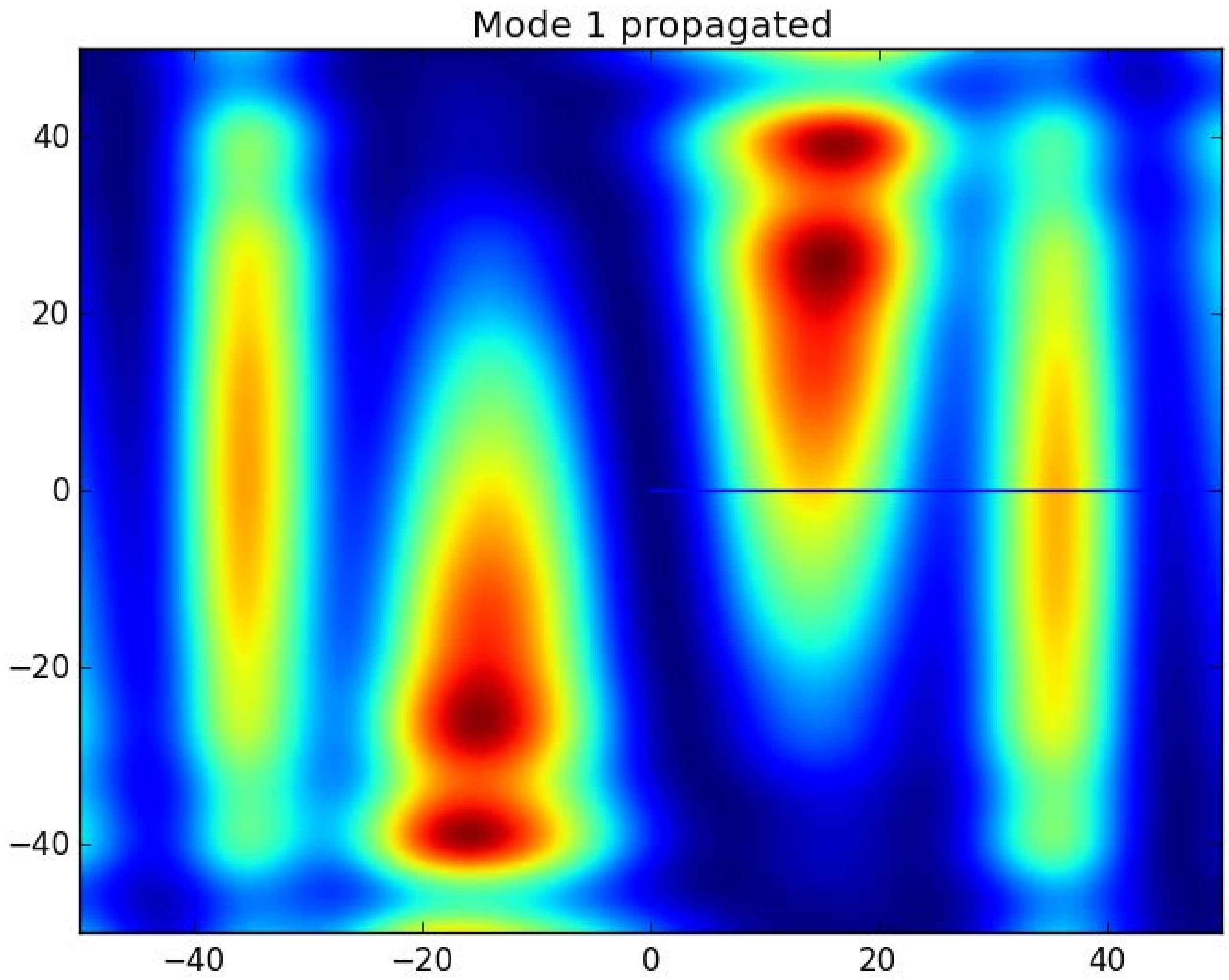
NUMERICS: INTENSITY FROM CONVOLUTION



NUMERICS: PROPAGATION OF MODES – CURRENT TRY



NUMERICS: PROPAGATION OF MODES – CURRENT TRY



High memory demands:
If the reference electric field E_0 is given on a grid of size n_x and n_y then memory requirement scale like $n_x^2 \cdot n_y^2$.

High CPU demands:
The numerical determination of Γ is computational expensive.
Diagonalization of big matrices is slow.

We do not really know yet if our results are converged.

OUTLOOK

In the near future:

- Propagate modes. Understand how SRW saves the wavefront and how an “external” wavefront can be propagated (quadratic phase term?)
- Do real tests. (for example: compare to SRW example 10)
- Check if the variation of the magnetic guide field across the electron beam dimension is really **negligible.**
- Incorporate energy spread.

In the future:

- Try to make an approximate desktop PC algorithm.(memory reduction)
- Alternatively the sets of modes could be saved and could be reused for every change in the beamline optics.

SUMMARY

- Using the brightness convolution theorem we calculate the mutual coherence function numerically. (**neglecting** variation of the magnetic guide field across the electron beam)
- We feed the convolution with a reference undulator electric field calculated with **SRW**.
- We perform a coherent mode decomposition and keep only a limited number of modes.
- We want to propagate the modes using a wavefront propagation algorithm (preferable SRW).

- In the future we want to benchmark against SRW and verify that variation of the magnetic guide field across the electron beam can be neglected.
- The memory consummation is high and scales with N^4. We dream of reducing it to N^3. In that case we would have a desktop problem.

Thank you for your attention